The Algorithmics of Solitaire-Like Games

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Algorithmic Problem Solving

- 1st-year, 1st-semester
- compulsory since 2006, previously optional
- problem-driven
- introduction to mathematics of program construction

The Bridge Problem

4 people want to cross a bridge.

Person 1 takes 1 minute to cross the bridge.
Person 2 takes 2 minute to cross the bridge.
Person 3 takes 5 minute to cross the bridge.
Person 4 takes 10 minute to cross the bridge.

The bridge is narrow and only 2 people can cross together. It is dark and they have only one torch. When someone is on the bridge they must have the torch with them.

Show that everyone can cross in 17 minutes.

The Bridge Problem

N people want to cross a bridge.

Person i takes t.i minute to cross the bridge.

The bridge is narrow and only 2 people can cross together. It is dark and they have only one torch. When someone is on the bridge they must have the torch with them.

Develop an algorithm to get all people across in the shortest possible time.

Seven-trees In One

(unlabelled) binary trees:

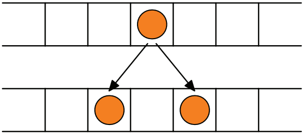
$$\mathbf{T} \cong \mathbf{1} + \mathbf{T}^2$$

generates

$$\mathbf{T}^7 \cong \mathbf{T}^1$$

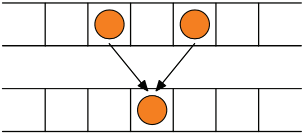
Seven-trees In One

fission



$T^k \times T^1 \rightarrow T^k \times (T^0 + T^2)$

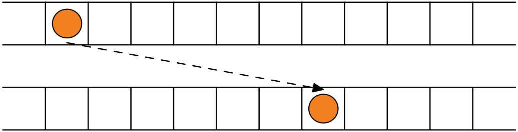
fusion



$T^k \times (T^0 + T^2) \rightarrow T^k \times T^1$

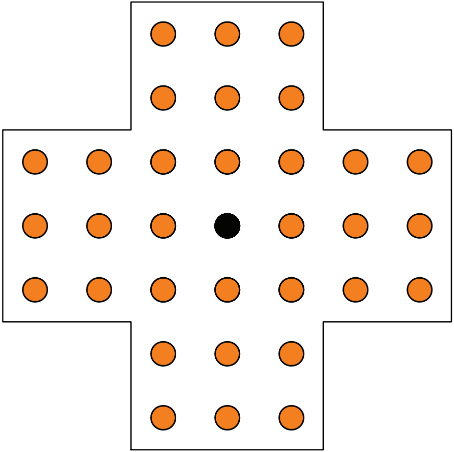
goal

T^1

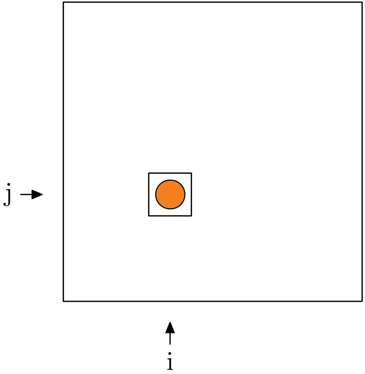


T^7

Peg Solitaire



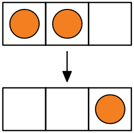
Weight of a Peg



Weight : p^{i+j}

Total weight = sum of weights of all pegs

Peg Solitaire



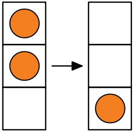
horizontal-right move

$$p^{i+j} + p^{i+j+1} \rightarrow p^{i+j+2}$$

$$p^{i+j} \times (p^0 + p^1) \rightarrow p^{i+j} \times p^2$$

$$p^0 + p^1 \rightarrow p^2$$

Peg Solitaire



vertical-down move

$$p^{i+j+2} + p^{i+j+1} \longrightarrow p^{i+j+0}$$

$$p^2 + p^1 \longrightarrow p^0$$

The Total Weight is invariant if

horizontal-right
vertical-up

$$p^0 + p^1 = p^2$$

horizontal-left
vertical-down

$$p^0 = p^1 + p^2$$

GF(4)

Semi-ring (in fact, a field) with 4 elements $\{0, 1, p, p^2\}$

$+$	0	1	p	p^2
0	0	1	p	p^2
1	1	0	p^2	p
p	p	p^2	0	1
p^2	p^2	p	1	0

Conclusion:

Weight in $GF(4)$ is invariant under all moves.

Hence positions are divided into 4 equivalence classes.

Symmetrically:

Assign weight p^{i-j} to peg at position (i, j).

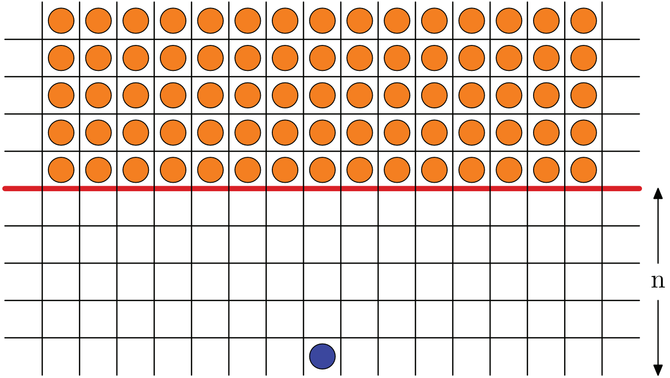
Then positions are divided into 4 equivalence classes.

Corollary (De Bruijn 1972, Reiss 1857) :

Sixteen different equivalence classes.

Solitaire Army (Conway, 1961)

A number of Solitaire men stand initially on one side of a straight line beyond which is an infinite empty desert. How many men do we need to send a scout just 0, 1, 2, 3, 4 or 5 paces out into the desert?



Monovariant ("Pagoda Function")

A *monovariant* is a function from the state space to a partially ordered set that is monotonic with respect to the transition relation.

(I.e. $s \rightarrow t \;\Rightarrow\; f.s \leqslant f.t$.)

Solitaire Army

Label position to be reached $(0, 0)$. Suppose Solitaire men are all initially at positions (i, j) such that $n \leqslant j$.

Assign to peg at position (i, j) the weight σ^{i+j}.

Choose σ so that the total weight is a monovariant.

$$\sigma^2 + \sigma = 1, \qquad \sigma^2 \leqslant 1 + \sigma$$

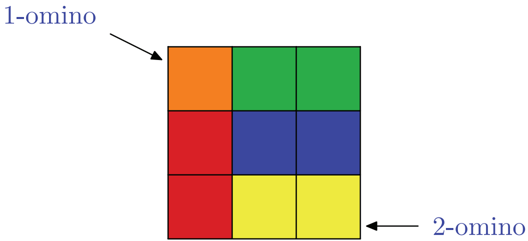
$$\sigma = \frac{\sqrt{5} - 1}{2} .$$

$$\left.\begin{array}{l}\text{Maximum initial weight} < \sigma^{n-5} , \\ \text{Goal state has weight} \geqslant 1 .\end{array}\right\} \text{Impossible for } 5 \leqslant n .$$

Tiling Problem

Given an $m \times m$ board, one 1-omino and an unlimited supply of n-ominoes, when is it possible to completely tile the board?

Example: $m = 3, \ n = 2$

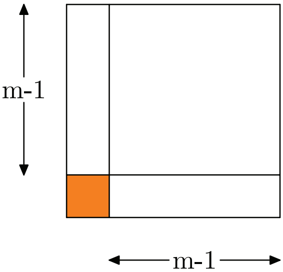


Tiling Problem

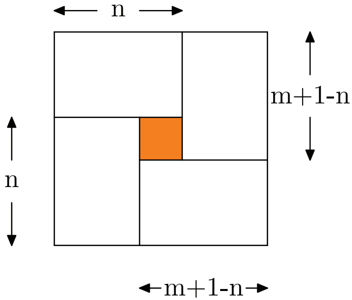
Solution:

$$1 = m \vee (1 \leqslant n < m \wedge (n \setminus (m-1) \vee n \setminus (m+1)))\ .$$

Sufficient:



$n \setminus m-1$



$n \setminus m+1$

Tiling Problem

Necessary:

Let $R = (A, 0, 1, +, \cdot)$ be any semiring with an element x s.t.

$$\langle \Sigma\, i : 0 \leqslant i < n : x^i \rangle =_R 0 \ .$$

Assign to square (i, j) the weight

$$\begin{array}{ll} x^{i+j} & \text{if the square is covered ;} \\ 0 & \text{otherwise .} \end{array}$$

The total weight is invariant under placement of an n-omino on the board.

Tiling Problem

The total weight is invariant under placement
of an n-omino on the board.

Necessary condition :

$$\langle \exists k :: x^k =_R \langle \Sigma i, j : 0 \leqslant i, j < m : x^{i+j} \rangle \rangle .$$

$\uparrow$ weight of board with one 1-omino (initial state)

$\uparrow$ weight of totally covered board (final state)

Tiling Problem

Choice of semiring R :

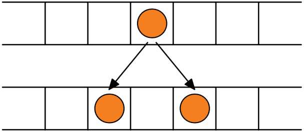
$$R \;=\; GF(2)[x]/\langle\, \Sigma\, i \,:\, 0 \leqslant i < n \,:\, x^i \,\rangle$$

(set of polynomials in indeterminate x with coefficients in $GF(2)$ modulo $\langle\, \Sigma\, i \,:\, 0 \leqslant i < n \,:\, x^i \,\rangle$)

$$\begin{array}{ll} & \langle\, \exists\, k \,::\, x^k =_R \langle\, \Sigma\, i, j \,:\, 0 \leqslant i, j < m \,:\, x^{i+j} \,\rangle \\ \Rightarrow & \{\, P =_R x^k \;\Rightarrow\; \#P = 1 \vee \#P = n-1 \,\} \\ & (m \mod n = 1) \vee (m \mod n = n-1) \end{array}$$

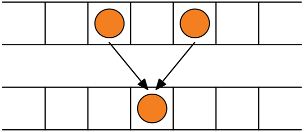
Seven-trees In One

fission



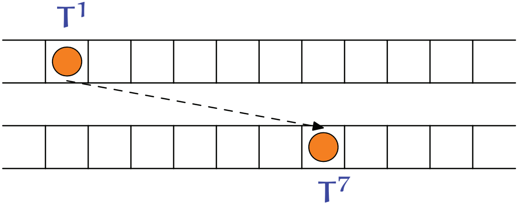
$$T^k \times T^1 \rightarrow T^k \times (T^0 + T^2)$$

fusion

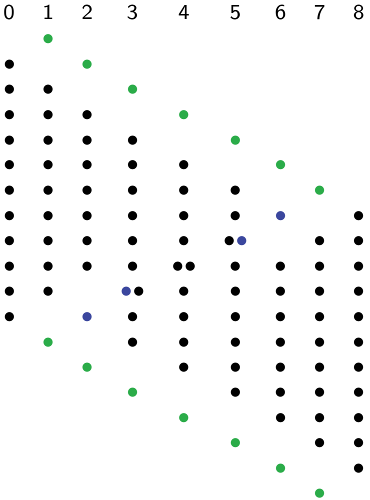


$$T^k \times (T^0 + T^2) \rightarrow T^k \times T^1$$

goal



Seven-trees In One



$$T^1+(\gamma+T^4+T^5)\times(1-T+T^2) = T^7+(\gamma+T^1+T^2)\times(1-T+T^2)$$

Cyclotomic constraints:

$$\mathbf{T} \cong \mathbf{T} + \boldsymbol{\Psi}$$

- $\boldsymbol{\Psi}$ is a product of cyclotomic polynomials;
- $\boldsymbol{\Psi}$'s degree is at least 2;
- coefficients of $\mathbf{T} + \boldsymbol{\Psi}$ are positive.

$$T^1 - 1 = (T - 1) \times (T + 1) \times (T^2 + T + 1) \times (T^1 - T + 1)$$

$$\begin{aligned}
\mathbf{T} &\cong \mathbf{1} + \mathbf{T}^2 & \{T^1 - T + 1\}\ ; \\
\mathbf{T} &\cong \mathbf{1} + \mathbf{2T} + \mathbf{T}^2 & \{T^2 + T + 1\}\ ; \\
\mathbf{T} &\cong \mathbf{1} + \mathbf{T} + \mathbf{T}^2 + \mathbf{T}^4 & \{(T^2 + T + 1)(T^1 - T + 1)\}\ .
\end{aligned}$$

Cyclotomic Games

$$\Psi_{a,n} = \langle \Sigma\, i : 0 \leqslant i < a : T^{i \times a^{n-1}} \rangle$$

$$(2 \leqslant a \wedge 2 \leqslant n) \ \vee \ (3 \leqslant a \wedge 1 = n)$$

— product of cyclotomic polynomials of degree at least 2.

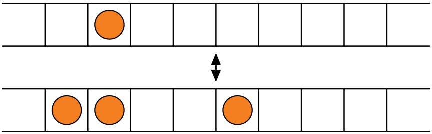
Cyclotomic Games

move : $\mathsf{T} = \mathsf{T} + \Psi_{a,n}$

goal : $\mathsf{T}^{a^n+1} - \mathsf{T} = (\mathsf{T}^{a^{n-1}+1} - \mathsf{T}) \times \Psi_{a,n}$

Example : $\Psi_{2,3} = 1 + \mathsf{T}^4$

move :



goal : move one checker 8 places to right.

Algorithm

$\{ s = T \}$

expand

$\{ s = T + (\gamma + T^{a^{n-1}+1}) \times \Psi_{a,n} \}$

$\{ s = T^{a^n+1} + (\gamma + T) \times \Psi_{a,n} \}$

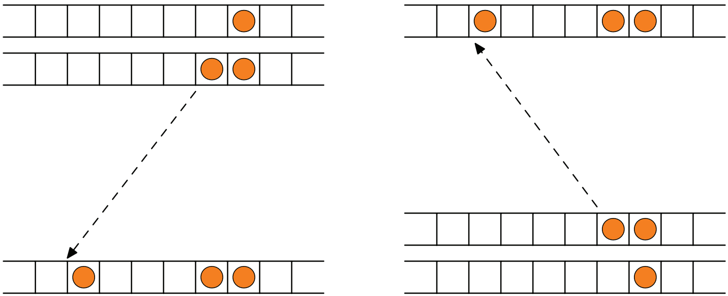
contract

$\{ s = T^{a^n+1} \}$

Contract

note: $\Psi_{a,n} \;=\; 1 + \ldots$

an expansion always adds a checker one place to the left of expanded checker.



Contract

$\{\, s = T^{a^n+1} + \langle\, \Sigma\, i \,:\, 0 \leqslant i < a^n + 1 \,:\, T^i \,\rangle \times \Psi_{a,n} \,\}$

k := 0;

$\{$ Invariant : $s = T^{a^n+1} + \langle\, \Sigma\, k \,:\, k \leqslant i < a^n + 1 \,:\, T^i \,\rangle \times \Psi_{a,n} \,\}$

do $k < a^n + 1 \;\rightarrow\; s, k \;:=\; s - T^k \times \Psi_{a,n},\, k + 1$

od

$\{\, s = T^{a^n+1} \,\}$

Expand

Let $E = \{ i : 2 \leqslant i < a^n + 1 : i + 1 \} \uplus \{ a^{n-1} + 2 \}$

$\{ s = T + \Psi_{a,n} \}$

$A, B \ := \ E, \emptyset;$

$\{ \text{Invariant} : s = T + \langle \Sigma\ i : i \in \{1\} \uplus B : T^{i-1} \rangle \times \Psi_{a,n}$

$\quad \wedge \ A \uplus B = E \}$

do $A \neq \emptyset \ \rightarrow$ choose $j \in A$ such that there is a checker in position j ;

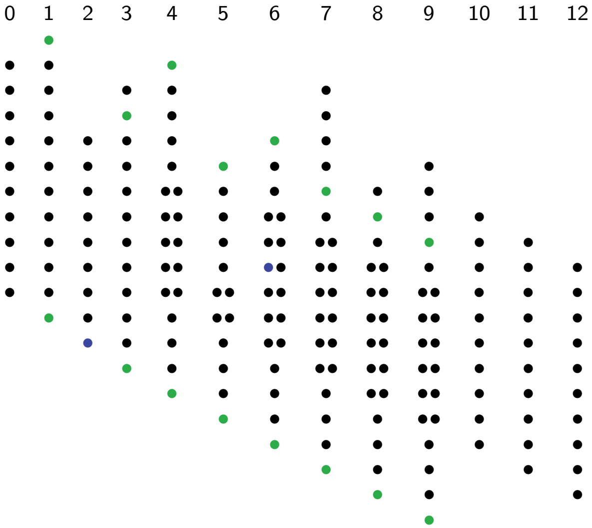
$s \ := \ s + T^{j-1} \times \Psi_{a,n}$;

$A, B \ := \ A - \{j\}, B \uplus \{j\}$;

od

$\{ s = T + \langle \Sigma\ i : i \in \{1\} \uplus E : T^{i-1} \rangle \times \Psi_{a,n} \}$

Solution For Example



$$T^1 + (\gamma + T^5) \times (1 + T^4) = T^9 + (\gamma + T^1) \times (1 + T^4)$$

Conclusion

- Goal is to develop exercises in algorithmic problem solving
- non-mathematical problem statement
- mathematical solution